Polynomial Kernels for λ-extendible Properties Parameterized Above the Poljak-Turzík Bound

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2. for every instance (G, k) of the Π-Subgraph Problem APTB, there exists $S \subseteq V(G)$ such that $|S| \leq \frac{6k}{1-\lambda}$ and $G - S$ is a forest of cliques.

Results

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2. If $\lambda = \frac{1}{2}$ and Π is an hereditary strongly λ-extendible property on simple graphs, the Π-Subgraph Problem APTB admits a polynomial kernel.

The structure of G

Let $\mathcal{B}$ denote the set of blocks of $G - S$.

$$\left.\begin{array}{c} |\mathcal{B}| \leq \widetilde{p}_1(k) \\ \textit{and} \\ \forall B \in \mathcal{B}, |B| \leq \widetilde{p}_2(k) \end{array}\right\} \Rightarrow \text{Kernel!}$$

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$\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_{\geq 3}$.

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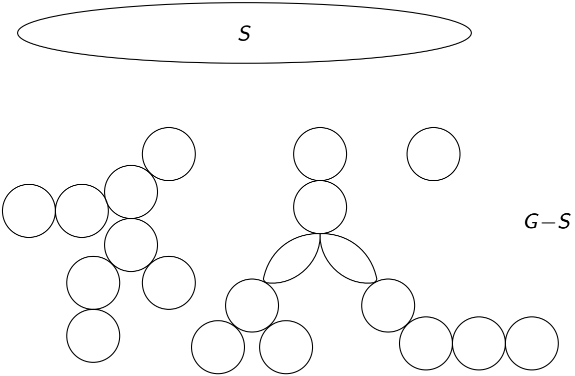

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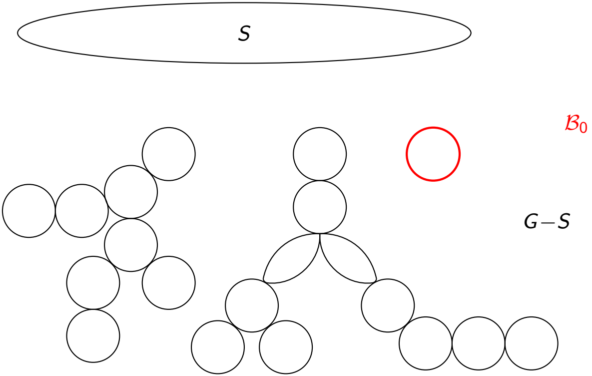

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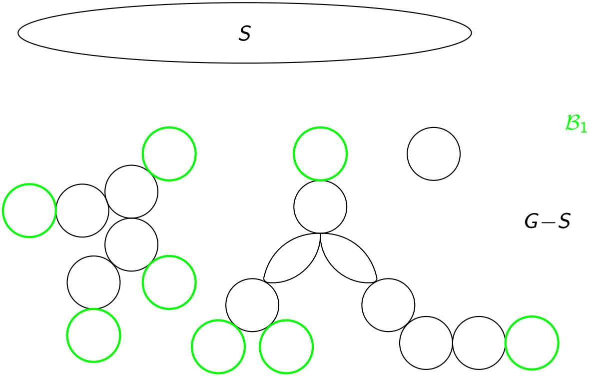

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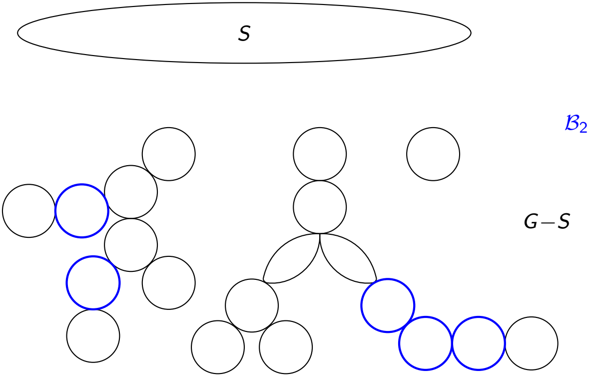

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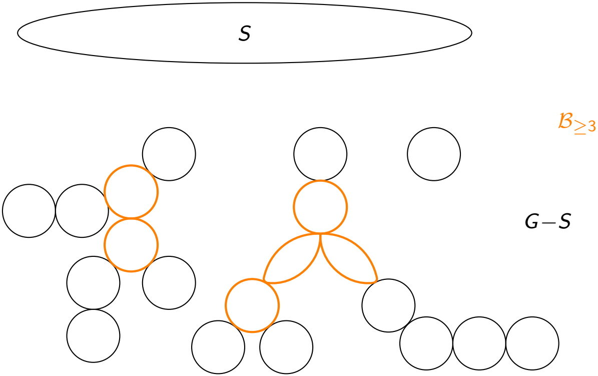

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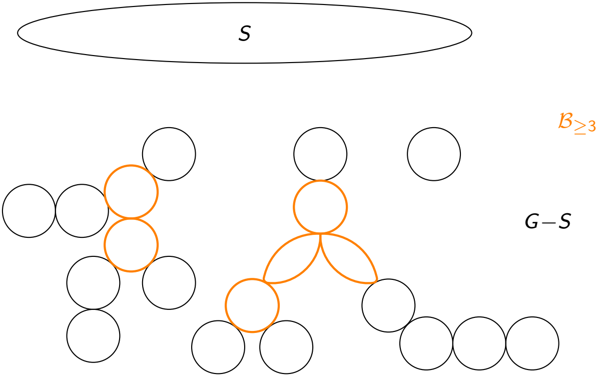

Lemma

For every forest of cliques $G - S$, $|\mathcal{B}_{\geq 3}| \leq 3|\mathcal{B}_1|$.

Definition (Diverges on cliques)

A strongly λ-extendible property *diverges on cliques* if there exists $j \in \mathbb{N}$ such that $ex(K_j) > \frac{1-\lambda}{2}$.

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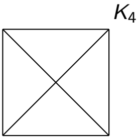
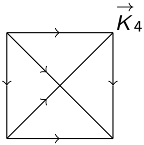


Max Cut



Max Acyclic Subgraph

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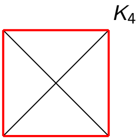

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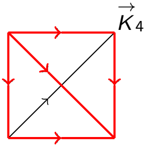

Max Acyclic Subgraph

Theorem

Let Π *be a strongly* λ*-extendible property which diverges on cliques. Then* $\lim_{s\to+\infty} ex(K_s) = +\infty$.

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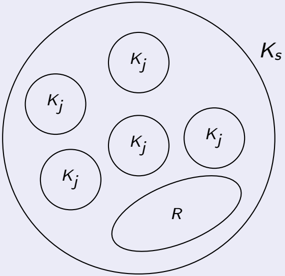

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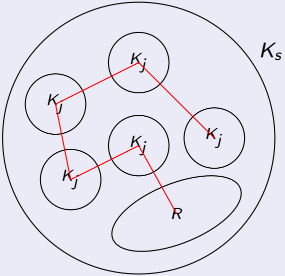

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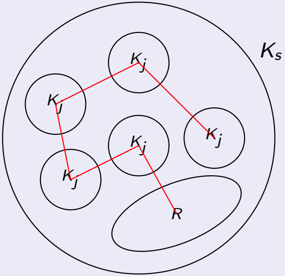

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$$\Downarrow$$

$$ex(K_s)\geq ca$$

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Bound on the block interior

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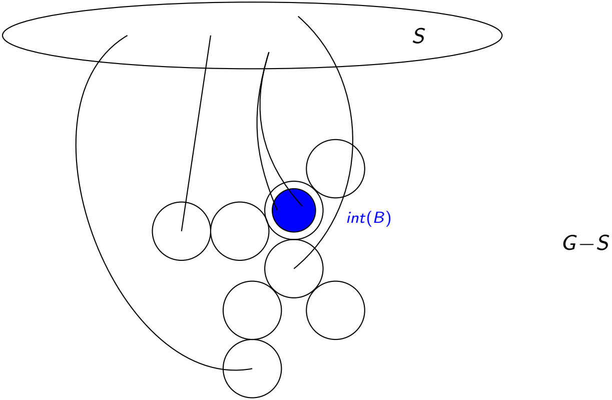

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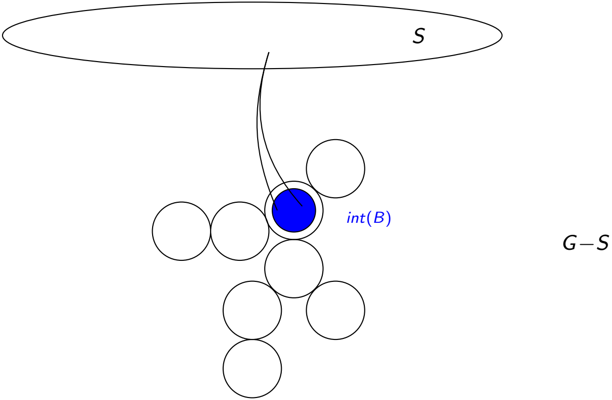

Bound on leaves

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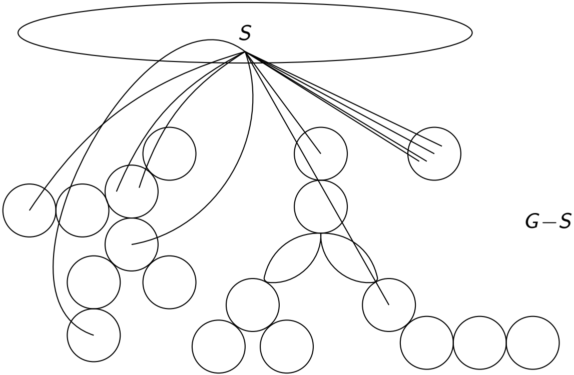
If Π diverges on cliques, there exists $r_2 = p_2(\Pi, \lambda, k)$ such that for every $s \in S$, $\sum_{B \in \mathcal{B}} |N_G(int(B)) \cap \{s\}| \geq r_2$ ensures that the answer is YES.



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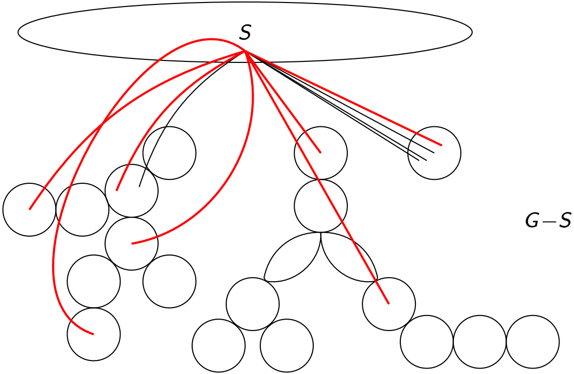

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If Π diverges on cliques and if a block path contains at least $r_3 = p_3(\Pi, \lambda, k)$ blocks with positive excess, then the answer is YES.

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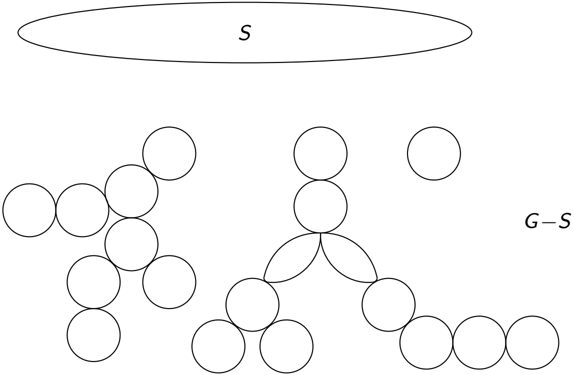

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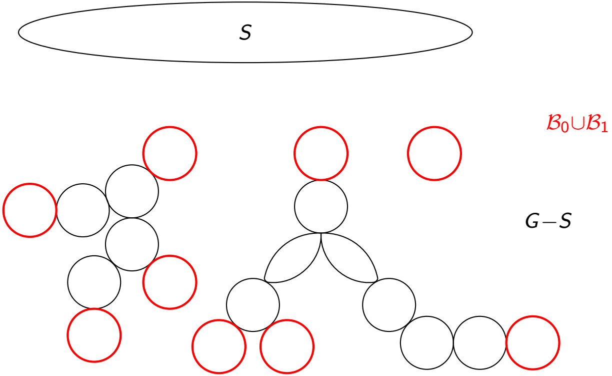

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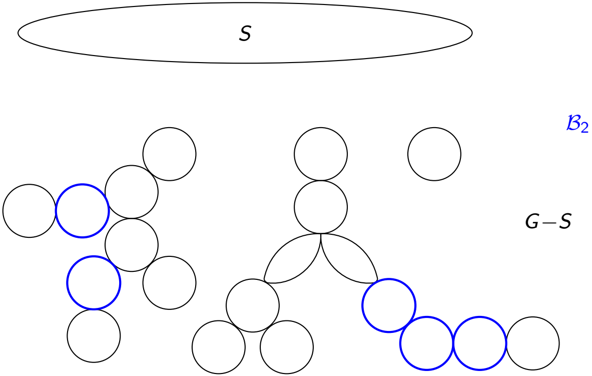

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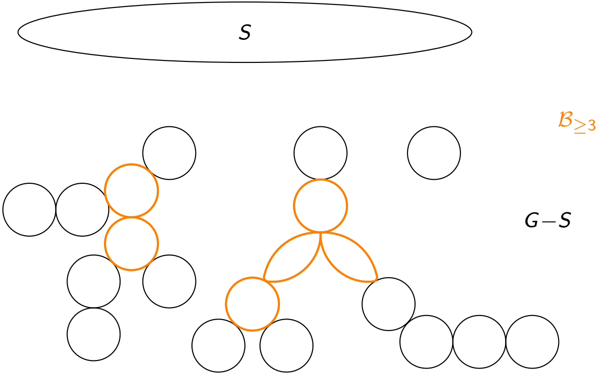

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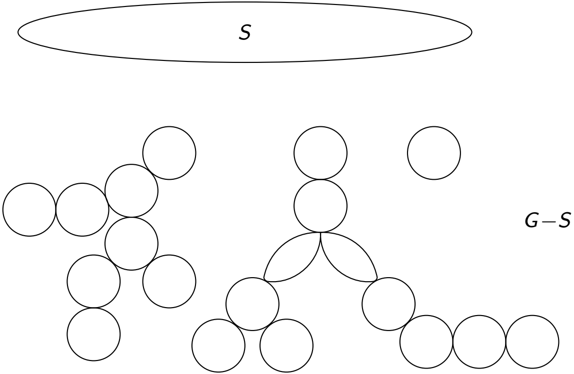

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If Π diverges on cliques, then $ex(K_3) > \frac{1-\lambda}{2}$ ($\Rightarrow$ Kernel).

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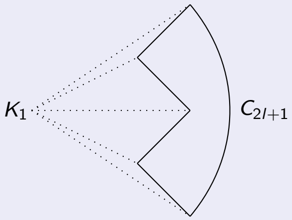

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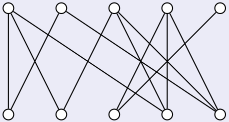

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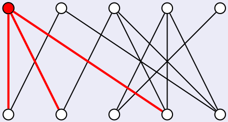

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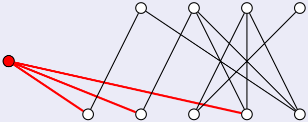

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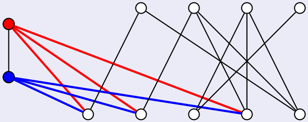

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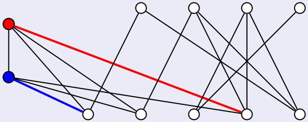

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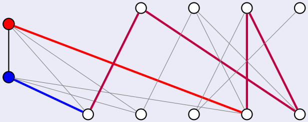

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- Kernel proof for weighted case.

Thanks!