Polynomial Kernels for λ-extendible Properties Parameterized Above the Poljak-Turzík Bound (Part 1)

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Geevarghese Philip Ashutosh Rai Saket Saurabh

WorKer 2013

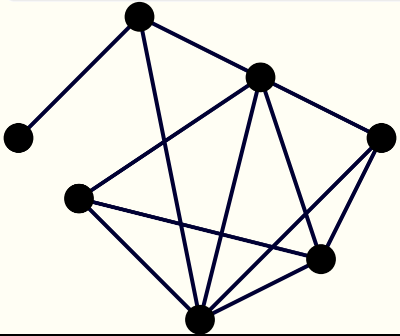
Max Cut

Problem (Max Cut)

Instance : *Graph G, integer p.*
Question: Does G have a bipartite subgraph with $\geq p$ edges?

Problem (Weighted Max Cut)

Instance : *Graph G with integer weights on edges, integer p.*
Question: Does G have a bipartite subgraph with weight $\geq p$?



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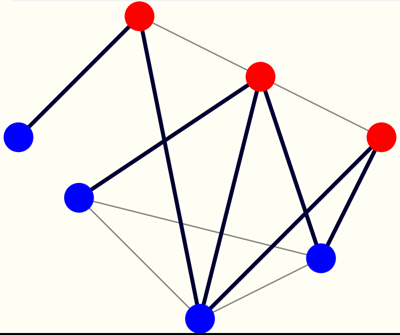

Max Cut Lower Bound

Theorem (Edwards-Edős Bound, 1973)

Let G be a connected graph with n vertices, m edges. Then G contains a bipartite subgraph with at least

$$\frac{m}{2} + \frac{n-1}{4}$$

edges.

Theorem

Let G be a connected graph with weight function $w : E(G) \to \mathbb{N}$. Then G contains a bipartite subgraph with weight at least

$$\frac{w(G)}{2} + \frac{w(T)}{4}$$

where T is a minimum weight spanning tree of G.

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λ-extendible properties

- Poljak and Turzík (1986) introduce the concept of λ-**extendible properties**, where $0 < \lambda < 1$.
- Extends notion of bipartiteness: being bipartite is a $\frac{1}{2}$-*extendible property*.
- A number of other graph properties are λ-extendible for some λ.
- For our proofs we use the slightly stronger concept of **strongly** λ-**extendible properties** (Mnich, Philip, Saurabh, Suchý, 2012)

λ-extendible properties

A graph property Π is strongly λ-extendible $(0 < \lambda < 1)$ if it satisfies the following conditions:

- **1. Inclusiveness:** $K_1, K_2 \in \Pi$.

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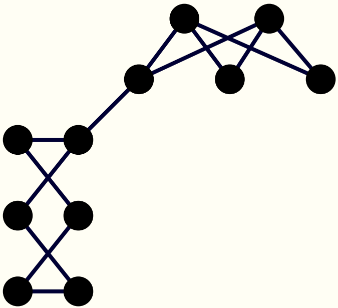
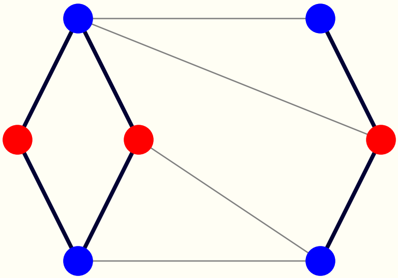
- **2. Block Additivity:** $G \in \Pi$ iff every block of G belongs to Π.



A graph property Π is strongly λ-extendible ($0 < \lambda < 1$) if it satisfies the following conditions:

- **3. Strong λ-Subgraph Extension:** Let (U, W) be a partition of $V(G)$ such that $G[U], G[W] \in \Pi$. Then there exists $F \subseteq E(U, W)$ such that $w(F) \geq \lambda w(E(U, W))$, and $G[U]$ and $G[W]$ together with the edges in F is a graph in Π.



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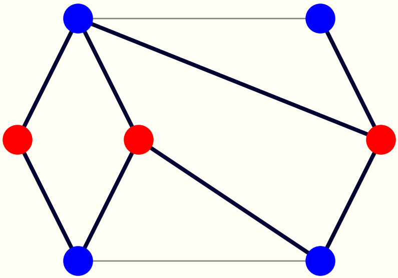

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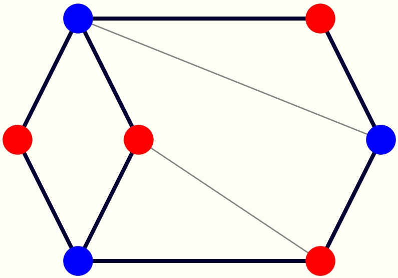

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A graph property Π is strongly λ-extendible ($0 < \lambda < 1$) if it satisfies the following three conditions:

- **1. Inclusiveness:** $K_1, K_2 \in \Pi$.
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- **3. Strong λ-Subgraph Extension:** Let (U, W) be a partition of $V(G)$ such that $G[U], G[W] \in \Pi$. Then there exists $F \subseteq E(U, W)$ such that $w(F) \geq \lambda w(E(U, W))$, and $G[U]$ and $G[W]$ together with the edges in F is a graph in Π.

(Normal λ-extendibility (as defined by Poljak, Turzík) may assume that $G[U] = K_2$ in condition 3.)
(Strong) λ-extendibility is also defined for **oriented graphs** and **edge-labelled graphs**.

Poljak-Turzík Bound

Theorem (Poljak-Turzík Bound, 1986)

Let G be a connected graph with weight function $w : E(G) \rightarrow \mathbb{N}$, and let Π be a λ-extendible property. Then G contains a subgraph with property Π with weight at least

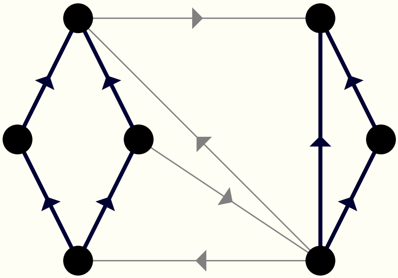
$$\lambda w(G) + \frac{(1-\lambda)w(T)}{2}$$

where T is a minimum weight spanning tree of G.

This result generalises the Edwards-Erdős Bound.

Example: Acyclic Graph

- In oriented graphs, the property of being acyclic is strongly $\frac{1}{2}$-extendible.



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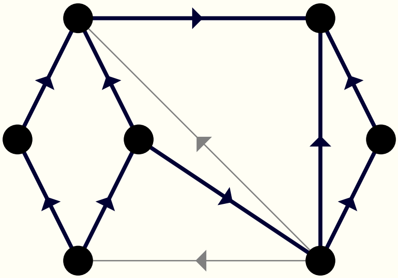

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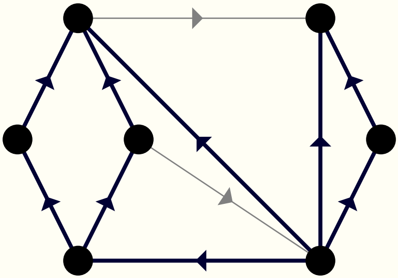


Example: Acyclic Graph

Corollary

Every oriented graph D has an acyclic subgraph with weight at least

$$\lambda w(D) + \frac{(1-\lambda)w(T)}{2}$$

$$= \frac{w(D)}{2} + \frac{w(T)}{4}$$

where T is a minimum weight spanning tree of D.

Example: Balanced Graph

- A **signed** graph is a graph in which all edges are labelled $+$ or $-$.
- A signed graph G is **balanced** if there is a partition (U, W) of $V(G)$ such that all edges in $E(U, W)$ are $-$ and all edges in $G[U]$, $G[W]$ are $+$.
- In signed graphs, the property of being a balanced graph is strongly $\frac{1}{2}$-extendible.

Corollary

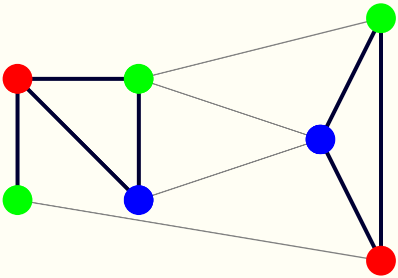
Every signed graph G has a balanced subgraph with weight at least

$$\frac{w(G)}{2} + \frac{w(T)}{4}$$

where T is a minimum weight spanning tree of G.

Example: r-Colorable Graph

- The property of being an r-colorable graph is strongly $\frac{r-1}{r}$-extendible.



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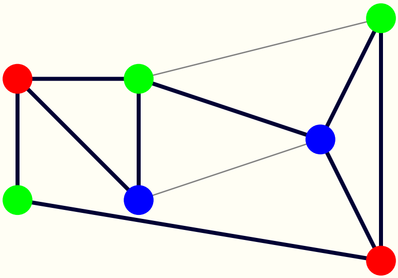

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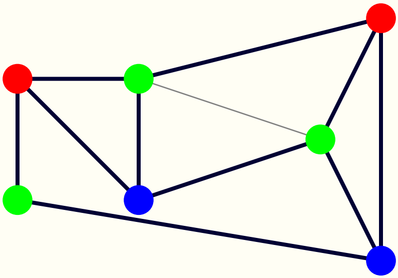

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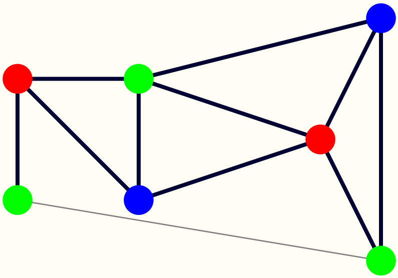

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Corollary

Every graph G has an r-colorable subgraph with weight at least

$$\frac{(r-1)w(G)}{r} + \frac{w(T)}{2r}$$

where T is a minimum weight spanning tree of G.

λ-Extendible Properties Parameterized Above PT

Given a (strongly) λ-extendible property Π, define:

Problem (Π-Subgraph Above Poljak-Turzík Bound (Π-Subgraph APTB))

Instance : Graph G with integer weights on edges
Parameter: Integer k.
Question: Does G have a subgraph in Π with weight at least

$$\lambda w(G) + \frac{(1-\lambda)w(T)}{2} + k?$$

Max Cut Parameterized Above EE

Problem (Max Cut Above Edwards-Erdős (Max Cut AEE))

Instance : *Graph G with integer weights on edges, integer k.*
Question: *Does G have a bipartite subgraph with weight at least*

$$\frac{w(G)}{2} + \frac{w(T)}{4} + k?$$

- Mahajan and Raman (1999) asked: Is unweighted MAX CUT AEE fixed-parameter tractable?

First FPT result

Theorem (Crowston,Mnich,Jones, 2012)

Unweighted MAX CUT AEE *has:*

- *FPT algorithm with running time* $O^*(2^{12k})$.
- *Kernel with* $O(k^5)$ *vertices.*

Proof idea: In polynomial time, either prove YES or find $S \subseteq V(G)$ such that:

- $|S| \leq 12k$
- $G - S$ is a simple kind of graph called a **forest of cliques**.

After guessing a partition on S, the problem can be solved in polynomial time.

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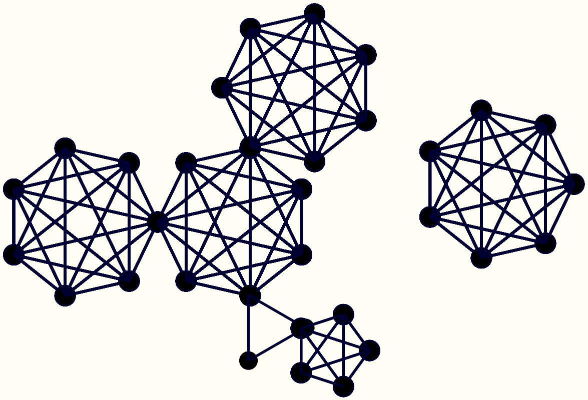
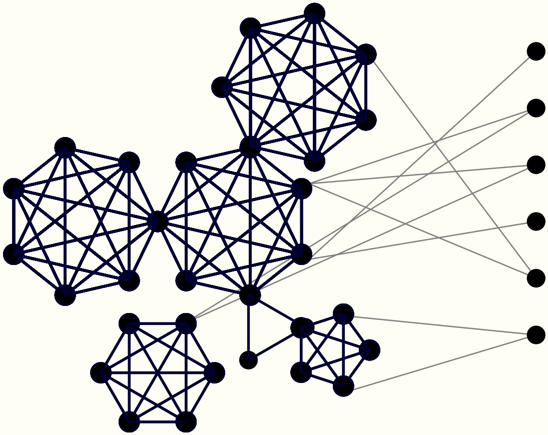
Forest Of Cliques

- **Forest of cliques:** A graph in which every block is a clique.



Almost-Forests Of Cliques

- Informally, if $S \subseteq V(G)$ is such that $|S| \leq f(k)$ and $G - S$ is a forest of cliques, we say G is an **almost-forest of cliques**.
- Unweighted Max Cut AEE is FPT on almost-forests of cliques.



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Theorem (Crowston,Mnich,Jones, 2012)

Unweighted MAX CUT AEE *has:*

- *FPT algorithm with running time* $O^*(2^{12k})$.
- *Kernel with* $O(k^5)$ *vertices.*

- What about weighted case?
- What about other λ-extendible properties?

FPT For Unweighted λ-Extendible Properties

Theorem (Mnich, Philip, Saurabh, Suchý, 2012)

Let Π be a **strongly** *λ-extendible property. Then unweighted* Π-SUBGRAPH ABOVE PT *is FPT, as long as it is FPT on almost-forests of cliques.*

- Proof: Adapts reduction to almost-forest of cliques used in MAX CUT result.
- This result is used to prove FPT for Unweighted ACYCLIC and r-COLORABLE SUBGRAPH problems above PT (and others).

Other results

Theorem (Crowston, Gutin, Jones, 2012)

Unweighted ACYCLIC SUBGRAPH APTB *has a kernel with* $O(k^2)$ *vertices.*

Theorem (Crowston, Gutin, Jones, Muciaccia, 2012)

Unweighted BALANCED SUBGRAPH APTB *is FPT and has a kernel with* $O(k^3)$ *vertices.*

Results So Far

- FPT for unweighted case provided FPT on almost-forests of cliques.
- Polynomial kernel for unweighted versions of specific problems.
- Still unknown: Anything about weighted case.
- Kernel for general λ-extendible properties?
- **This talk:** extension of general FPT proof to weighted case.
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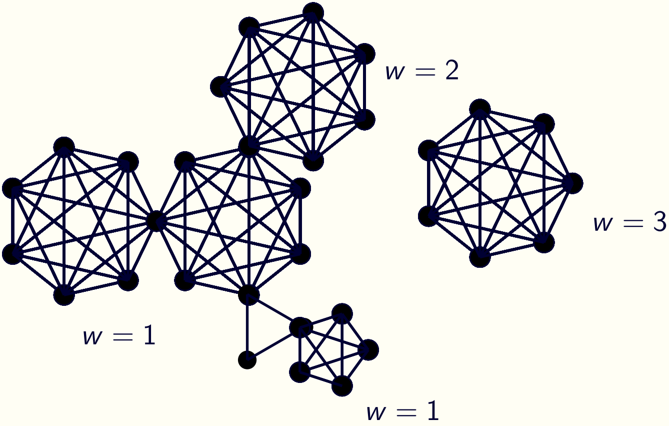
Forest of uniform cliques

- A **forest of uniform cliques** is a forest of cliques in which all edges in a block have the same weight.



Weighted FPT Result

Theorem (Crowston, Jones, Muciaccia, Philip, Rai, Saurabh, 2013)

Let Π be a strongly λ-extendible property. Then Π-Subgraph APTB *is FPT, as long as it is FPT on almost-forests of* **uniform** *cliques.*

In polynomial time, we will either prove Yes or find $S \subseteq V(G)$ such that

- $|S| \leq \frac{6k}{1-\lambda}$
- $G - S$ is a forest of uniform cliques.

The theorem then follows.

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Finding S

- How do we find S?
- Use 'One-way reduction rules'.
- Assume something about the solution; reduce k and delete a few vertices.
- If we reduce to an instance with parameter ≤ 0, answer is YES.
- Otherwise, our assumptions about the solution were wrong.
- **But** in this case we deleted a few vertices to get a graph for which no reduction rules apply.
- Let S be the deleted vertices; it will turn out that $G - S$ is a forest of uniform cliques.

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Reduction Rule 1

- Suppose $\exists$ adjacent edges ab, bc s.t $w(ab) > w(bc)$ and $G - \{a, b\}$ connected.
- We assume edge ab is in solution (and bc is part of minimum spanning tree).
- Delete a, b and reduce k by $\frac{(1-\lambda)}{2}(w(ab) - w(bc))$

Reduction Rule 2

- Suppose $\exists$ induced P_3 abc s.t. $G - \{a, b, c\}$ connected
- (and not all edges between $G - \{a, b, c\}$ and $\{a, b, c\}$ are heavier than $w(ab)$).
- Assume $G[\{a, b, c\}]$ is in solution.
- Delete a, b, c and reduce k by $\frac{(1-\lambda)}{2} w(ab)$.

Reduction Rule 3

- Suppose G has a cut vertex.
- Apply reduction rules to each block in turn.

Finding S

- Our reduction rules:
  1. If $\exists$ adjacent egdes ab, bc s.t $w(ab) > w(bc)$ and $G - \{a, b\}$ connected, **assume** ab **in solution**, delete a, b and reduce k by $\frac{(1-\lambda)}{2}(w(ab) - w(bc))$.
  2. If $\exists$ induced P_3 abc s.t. $G - \{a, b, c\}$ connected (and not all edges between $G - \{a, b, c\}$ and $\{a, b, c\}$ are heavier than $w(ab)$), **assume** $G[\{a, b, c\}]$ **in solution**, delete a, b, c and reduce k by $\frac{(1-\lambda)}{2}w(ab)$.
  3. If G has a cut vertex, apply rules to each block separately.
- Let S be the set of vertices deleted by rules 1, 2.

Lemma

For any connected graph G, either one of the reduction rules applies or G is a uniform-weighted clique.

- It follows that $G - S$ is a forest of uniform cliques.

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- $G - S$ is a forest of uniform cliques.
- For each rule we add at most 3 vertices to S and reduce k by at least $\frac{(1-\lambda)}{2}$.
- Therefore $\frac{|S|}{3} \leq \frac{2k}{1-\lambda}$ (or we have a YES-instance).
- Thus we have S with $|S| \leq \frac{6k}{1-\lambda}$ such that $G - S$ is a forest of uniform cliques.

Theorem (Crowston, Jones, Muciaccia, Philip, Rai, Saurabh, 2013)

Let Π be a strongly λ-extendible property. Then Π-SUBGRAPH APTB is FPT, as long as it is FPT on almost-forests of **uniform** *cliques.*

Conclusions

- Previous general result: unweighted case is FPT if FPT on almost-forests of cliques.
- New general results: weighted case is FPT if FPT on almost-forests of uniform cliques.
- Next talk: General polynomial kernel results for unweighted case.
- Open question: Kernel for weighted case?
- Open question: How to get rid of/decide 'FPT on almost-forests of cliques' condition?

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End Of Part 1

Thanks for your attention!