Matroid theory and kernelization

Part 2

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Introduction

Recap – matroids and gammoids

matroids:

- are $M = (E, \mathcal{I})$ with $\mathcal{I} \subseteq 2^E$ such that $\emptyset \in \mathcal{I}$ and ...
- arise from many natural examples: linear, transversal, graphic, ...
- may be representable by a matrix A with columns labeled by E such that $U \in \mathcal{I}$ iff the column vectors U are linearly independent
- e.g. all graphic matroids are representable by 0/1-matrices

Recap – matroids and gammoids

gammoids:

- arise from digraph $D = (V, A)$ with sources S and sinks T
  - $M = (T, \mathcal{I})$
  - $U \in \mathcal{I}$ iff exist $|U|$ disjoint paths from S to U
- are representable/linear matroids
  - efficient randomized algorithm gives matrix A with columns T
  - s.t. $U \in \mathcal{I}$ iff columns U are linearly independent ($A[., U]$ has rank $|U|$)

Outline

1. Using gammoids as a mincut “data structure”
2. The representative sets lemma
3. Relation of cuts and independent sets of gammoids
4. A randomized compression for the digraph pair cut problem
5. A randomized $\mathcal{O}(k^3)$ kernel for multiway cut with deletable terminals

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Answering basic flow questions via a gammoid

given: gammoid $M = (V, \mathcal{I})$ on digraph $D = (V, A)$ with sources S; also given a set of sinks $T \subseteq V$
want: value of maximum S,T-flow (number of disjoint S,T-paths)

- for analysis fix maximum S,T-path packing $\mathcal{P}$
- let $T^* \subseteq T$ be the endpoints of the paths in $\mathcal{P}$
- clearly T^* is an independent set in the gammoid
- start with $T' = \emptyset$
- add vertices from T to T' while preserving independence of T'
- while $|T'| < |T^*|$ by augmentation axiom we find $t \in T \setminus T' \supseteq T^* \setminus T'$ that can be added (note that $T \supseteq T^*$)
- we find independent set $T' \subseteq T$ of size $|T^*|$ (max flow value)

Answering more flow questions via a gammoid

given: digraph $D = (V, A)$ with terminals X
want: a gammoid with ground set of size polynomial in $|X|$ that gives maximum S, T-flow in $D - R$ for all partitions $X = S \cup T \cup R \cup U$.

- basic trick:
  - add set X' of vertices x' for all $x \in X$
  - add arcs (x', x) for all $x \in X$
  - let X' be the set of sources
- consequence:
  - adding x' to an independent set effectively removes the source x'
  - else it works the same as a source at x

Answering more flow questions via a gammoid II

given: partition $X = S \cup T \cup R \cup U$
want: set $I \subseteq X \cup X'$ that is independent iff there are $|T|$ vertex-disjoint paths from S to T in $D - R$

ensure effective deletion of $\mathbf{x} \in \mathbf{R}$:

- let $x \in I$ and $x' \notin I$
- block x from being used as internal vertex in other paths
- allows path (x', x) that does not block anything else

ensure that no paths start from $\mathbf{x} \in \mathbf{T} \cup \mathbf{U}$:

- let $x' \in I$ forcing a path (x') and effectively removing the source

Answering more flow questions via a gammoid III

given: partition $X = S \cup T \cup R \cup U$
want: set $I \subseteq X \cup X'$ that is independent iff there are $|T|$ vertex-disjoint paths from S to T in $D - R$

request nontrivial path to $\mathbf{x} \in \mathbf{T}$:

- let $x, x' \in I$
- this needs (x') and a path $(y', y, \ldots, x)$

ensure that paths can start in $\mathbf{x} \in \mathbf{S}$:

- let $x, x' \notin I$
- allows path $(x', x, \ldots, y)$

Answering more flow questions via a gammoid IV

given: partition $X = S \cup T \cup R \cup U$
want: set $I \subseteq X \cup X'$ that is independent iff there are $|T|$ vertex-disjoint paths from S to T in $D - R$

recap:

- make set X' of x' for all $x \in X$ and add arcs (x', x)
- take gammoid $M = (X \cup X', \mathcal{I})$ on this graph with sources X'
- to query for $|T|$ disjoint S, T-paths in $D - R$ check independence of I
- **exercise:** how to get max S, T-flow in $D - R$?
- **hint:** make correct choices to enforce S and R; then augment

mincut "data structure" – Wrap up

- gammoids naturally encode all S, T'-mincut values in D for fixed S
- by small modification of D we can get value of S, T-mincut in $D - R$ for all choices of $S \cup T \cup R \subseteq X$ from same gammoid
- ground set size is $2|X|$ allowing a small representation
- this suffices to get a randomized polynomial compression (+kernel) for Odd Cycle Transversal
  - need to know that OCT algorithm of Reed,Smith&Vetta comes down to mincut computations
  - could run similar algorithm to shrink approximate OCT solution $\tilde{X}$
  - all necessary mincut values can be read from auxiliary graph with $2|\tilde{X}|$ "terminals"
  - thus, gammoid for auxiliary graph suffices to represent the instance

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The representative sets lemma – preview

1. what are representative sets?
2. the representative sets lemma
3. dummy application of the lemma for Vertex Cover

Definition: Let $M = (E, \mathcal{I})$ be a matroid and let $X, Y \in \mathcal{I}$.

Then *X extends Y* (and vice versa) if

- $X \cap Y = \emptyset$ and
- $X \cup Y$ is independent, i.e., $X \cup Y \in \mathcal{I}$.

comment: could use $X, Y \in 2^E$ but $X \cup Y \in \mathcal{I}$ requires $X, Y \in \mathcal{I}$

Definition: Let $M = (E, \mathcal{I})$ be a matroid and let $\mathcal{Y} \subseteq \mathcal{I}$. Let $\mathcal{Y}^* \subseteq \mathcal{Y}$.

Then *$\mathcal{Y}^*$ is r-representative for $\mathcal{Y}$* if for every independent set $X \in \mathcal{I}$ of size at most r if some $Y \in \mathcal{Y}$ extends X then also some $Y^* \in \mathcal{Y}^*$ extends X.

comment: could use $\mathcal{Y} \subseteq 2^E$ but only $Y \in \mathcal{Y} \cap \mathcal{I}$ can extend anyway (and hence $\mathcal{Y} \cap \mathcal{I}$ is r-representative for $\mathcal{Y}$)

representative sets – combined definition & example

Definition: Let $M = (E, \mathcal{I})$ be a matroid and let $\mathcal{Y} \subseteq \mathcal{I}$. Let $\mathcal{Y}^* \subseteq \mathcal{Y}$.

Then $\mathcal{Y}^*$ is r-representative for $\mathcal{Y}$ if for every independent set $X \in \mathcal{I}$ of size at most r if there is $Y \in \mathcal{Y}$ with $X \cap Y = \emptyset$ and $X \cup Y \in \mathcal{I}$ then there is also $Y^* \in \mathcal{Y}^*$ with $X \cap Y^* = \emptyset$ and $X \cup Y^* \in \mathcal{I}$.

example:

- graph $G = (V, E)$, integer k
- $M = (V, 2^V)$ the free matroid, i.e. all subsets of V are independent

Q: when does an edge $\{u, v\}$ extend a set $X \subseteq V$?
A: if $\{u, v\} \cap X = \emptyset$ i.e. if X does not cover $\{u, v\}$

Q: what is the meaning of a k-representative subset E^* of E?
A: if any X, with $|X| \leq k$, does not cover some $\{u, v\} \in E$ then it also does not cover some $\{p, q\} \in E^*$

representative sets – example (cont.)

example:

- graph $G = (V, E)$, integer k
- $M = (V, 2^V)$ the free matroid, i.e. all subsets of V are independent

Q: what is the meaning of a k-representative subset E^* of E?
A: if any X, with $|X| \leq k$, does not cover some $\{u, v\} \in E$ then it also does not cover some $\{p, q\} \in E^*$

proposition: if E^* is k-representative for E then $(G = (V, E), k)$ is **yes** for Vertex Cover if and only if $(G' = (V, E^*), k)$ is **yes**.

Q: ok, so after deleting isolated vertices from G' we get a kernel if(!) we can find a small set E^*. how do we do that?
A: Ad hoc (like Buss' kernelization) or using a cool lemma!

the representative sets lemma

simple reformulation for this talk:

Lemma (Lovász, Marx): Let M be a represented matroid and let $\mathcal{Y}$ be a collection of independent sets, each of size s. We can efficiently find a set $\mathcal{Y}^*$ of size at most $\binom{r+s}{s}$ that is r-representative for $\mathcal{Y}$.

original lemma:

Lemma (Lovász, Marx): Let M be a represented matroid of rank $r + s$ and let $\mathcal{Y}$ be a collection of independent sets, each of size s.

- If $|\mathcal{Y}| > \binom{r+s}{s}$, then there is a set $Y \in \mathcal{Y}$ such that $\mathcal{Y} \setminus \{Y\}$ is r-representative for $\mathcal{Y}$.
- Furthermore, given a representation A of M, we can find such a set Y in $(|\mathcal{Y}| + ||A||)^{\mathcal{O}(1)}$ time.

example revisited

Lemma (Lovász, Marx): Let M be a represented matroid and let $\mathcal{Y}$ be a collection of independent sets, each of size s. We can efficiently find a set $\mathcal{Y}^*$ of size at most $\binom{r+s}{s}$ that is r-representative for $\mathcal{Y}$.

- graph $G = (V, E)$, integer k; $M = (V, 2^V)$ the free matroid
- the lemma provides a k-representative set $E^* \subseteq E$ for E
  - $|E^*| \leq \binom{k+2}{2} = \mathcal{O}(k^2)$
  - if k vertices avoid an edge of E then they also avoid an edge of E^*
- this gives a kernel with $\mathcal{O}(k^2)$ edges for Vertex Cover:
  - if $G' = (V, E^*)$ has a vertex cover X of size at most k
  - then X avoids no edge of E^*
  - then X can avoid no edge of E
  - hence $(G = (V, E), k)$ is **yes** too
  - converse holds trivially

Representative sets – wrap-up

- extending independent sets
- r-representative sets
- representative sets lemma
  - given a collection $\mathcal{Y}$ of sets of size at most s there is an r-representative subset $\mathcal{Y}^*$ of size $\binom{r+s}{s} = \mathcal{O}((r+s)^s)$
  - sets $Y^* \in \mathcal{Y}^*$ can extend any independent sets of size up to r that can be extended by any $Y \in \mathcal{Y}$
  - can find $\mathcal{Y}^*$ efficiently for any represented matroid (e.g. gammoids)
- using free matroid $M = (V, 2^V)$ a k-representative set of edges gives polynomial kernel for Vertex Cover

next: what can we get from the lemma by using gammoids instead of the free matroid?

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Relation of cuts and independent sets of gammoids

In this part of the talk we are interested in the following question:
Given a digraph $D = (V, A)$ with sources $S \subseteq V$.
What is the relation between

1. cuts separating S from parts of V and
2. independent sets of the gammoid $M = (V, \mathcal{I})$ on $D = (V, A)$ with sources S?

what if an S, T-cut is not independent?

let $\mathbf{X}$ be an $\mathbf{S, T}$-cut in $\mathbf{D = (V, A)}$ that is not independent

- then there are less than $|X|$ vertex-disjoint S, X-paths
- there is an S, X-cut X' of size less than $|X|$
- easy to see that X' is also an S, T-cut:
  - assume path P from S to T in $G - X'$
  - then P contains at least one vertex of X
  - this yields S, X-path avoiding X'; contradiction

in other words: if an S, T-cut X is not independent then there is a strictly cheaper S, T-cut that is also an S, X-cut

intuition: always better to cut closer to S

cutting as close to S as possible

Definition: A set X is closest (to S) if it is the unique minimum S, X-cut.

- clearly a closest set X must be independent, **else:** smaller S, X-cut
  (recall: independence means that we have $|X|$ paths from S to X)
- since we cannot cut any closer to S at the same cost ...
- ... all closer cuts must be bigger ...
- ... does that give more than $|X|$ disjoint S, X-paths?

note: to make sense of more than $|X|$ vertex-disjoint S, X-paths we must allow paths to share endpoints in X

intermission – sink-only copies of vertices

have: gammoid on digraph $D = (V, A)$ with sources S
want: clean way of formalizing that multiple paths share an endpoint

- for each $v \in V$ introduce one or more sink-only copies $v', v'', \ldots$:
  - for each arc (u, v) add (u, v')
  - for edge $\{u, v\}$ add arc (u, v')
  - ignore outgoing arcs (v, u) of v
- observe that
  - sink-only copies can only be endpoints of paths (no outgoing edges/arcs)
  - path ending in v could also end in v' and vice versa
- effectively this allows multiple paths ending in "same" vertex

cutting as close to S as possible (cont.)

Definition: A set X is closest (to S) if it is the unique minimum S, X-cut.

- clearly a closest set X must be independent, **else:** smaller S, X-cut
  (recall: independence means that we have $|X|$ paths from S to X)
- since we cannot cut any closer to S at the same cost ...
- ... all closer cuts must be bigger ...
- ... does that give more than $|X|$ disjoint S, X-paths?

proposition: a set X is closest to S if and only if $X \cup \{x'\}$ is independent, i.e. $|X| + 1$ vertex-disjoint paths from S to $X \cup \{x'\}$, for all $x \in X \setminus S$

proving the proposition I

proposition: a set X is closest to S if and only if $X \cup \{x'\}$ is independent, i.e. $|X| + 1$ vertex-disjoint paths from S to $X \cup \{x'\}$, for all $x \in X \setminus S$

proof (if):

- let $Z \neq X$ an S, X-cut of size at most $|X|$
- let $x \in X \setminus Z$, w.l.o.g.[1] $x \notin S$
- there are $|X| + 1$ vertex-disjoint paths from S to $X \cup \{x'\}$
- one of the paths avoids Z and reaches $X \cup \{x'\}$ (and hence X)
- contradiction; there can be no such alternative cut Z

[1] Z must contain $X \cap S$ to be S, X-cut

proving the proposition II

proposition: a set X is closest to S if and only if $X \cup \{x'\}$ is independent, i.e. $|X| + 1$ vertex-disjoint paths from S to $X \cup \{x'\}$, for all $x \in X \setminus S$

proof (only if):

- let $x \in X \setminus S$ such that $X \cup \{x'\}$ is not independent
- then there is a minimal $S, X \cup \{x'\}$-cut Z of size at most $|X|$
- if $x' \in Z$ then $Z - x'$ is S, X-cut smaller than X; done
- henceforth $x' \notin Z$
- if $x \in Z$ then exists S, x-path avoiding $Z - x$
  implies S, x'-path avoiding Z; contradiction
- thus $x \notin Z$ and hence $Z \neq X$ is different S, X-cut; done

what we learned so far...

1. in general S, T-cuts do not have to be independent sets
2. minimum S, T-cuts X are independent
3. cutting closer to S at the same cost never hurts
4. closest cuts/sets X are even "extra independent", i.e. $X \cup \{x'\}$ is independent for all $x \in X \setminus S$

remaining question: in analogy to our VERTEX COVER example, what does it mean to extend an independent set (that is a closest cut)?

extending closest sets

proposition: Let X be a set closest to S and let $v \in V \setminus X$. Then $X \cup \{v\}$ is independent if and only if v is reachable from S in $D - X$.

proof:

- if $X \cup \{v\}$ is independent then
  - have $|X| + 1$ paths from S to $X \cup \{v\}$
  - path from S to v avoids X and shows reachability in $D - X$
- if $X \cup \{v\}$ is not independent then
  - there is an $S, X \cup \{v\}$-cut Z with $|Z| \leq |X|$
  - Z is also S, X-cut of size at most $|X|$
  - hence $Z = X$ (recall that X is unique min S, X-cut)
  - thus X is an $S, X \cup \{v\}$-cut and v is not reachable from S in $D - X$

note: second part requires closeness, otherwise the non-independence of $X \cup \{v\}$ might be due to some earlier bottleneck

Cuts and independent sets – Wrap up

- minimum cuts are independent
- closest sets...
  - are "extra independent" sets
  - do not have to be cuts (other than separating themselves from S)
  - can replace any S, T-cut X by closest S, X-cut of at most same size
- extending a closest set is equivalent to being reachable

proposition: X is closest to $S \Leftrightarrow X \cup \{x'\}$ is independent for $x \in X \setminus S$.

proposition: Let X be a set closest to S and let $v \in V \setminus X$. Then $X \cup \{v\}$ is independent if and only if v is reachable from S in $D - X$.

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The digraph pair cut problem

Digraph Pair Cut (DPC)
Input: A directed graph $D = (V, A)$, source vertices $S \subseteq V$, a collection $P \subseteq \binom{V}{2}$ of sink vertices, and an integer k.
Output: Can one find a set X of at most k vertices such that in $D - X$ the set S cannot reach both vertices u and v of any pair $\{u, v\} \in P$?

generalizes Vertex Cover:

- let $(G = (V, E), k)$ an instance of VERTEX COVER
- digraph $D = (V, \emptyset)$, sources $S := V$, pairs $P := E$
- need to disconnect V from u or v for all $\{u, v\} \in P$ with at most k vertex deletions

note: say that "pair $\{u, v\}$ is reachable from S" if S reaches both u and v

Digraph Pair Cut

given: Digraph Pair Cut instance: digraph $D = (V, A)$, sources $S \subseteq V$, sink pairs $P \subseteq \binom{V}{2}$ and integer k
want: smaller, representative set of pairs $P^* \subseteq P$ such that (D, S, P^*, k) is an equivalent instance; ideally $|P^*| = k^{\mathcal{O}(1)}$

why do we want this?

- gammoid encoding of minimum cuts between S and subsets of $V(P^*)$
- cause it works for Vertex Cover

Using closest sets as solutions

- if X is solution for $D = (V, A)$ with sources $S \subseteq V$ and pairs $P \subseteq \binom{V}{2}$
- then so is any minimum S, X-cut X' (of size at most $|X|$):
  - if S could reach u and v in $D - X'$ for some $\{u, v\} \in P$
  - then one of the two paths must intersect X (as X is solution)
  - but then S reaches X in $D - X'$
- henceforth we consider only solutions X that are closest to S
- recall property of closest sets:

proposition: Let X be a set closest to S and let $v \in V \setminus X$. Then $X \cup \{v\}$ is independent if and only if v is reachable from S in $D - X$.

need: make this work regarding reachability of pairs

A simple gammoid for reachability of pairs

given: digraph $D = (V, A)$, sources $S \subseteq V$, sink pairs $P \subseteq \binom{V}{2}$
want: a gammoid M "where things work as needed" :-)

- let D' consist of two disjoint copies of D with $v \mapsto v(1), v(2)$
- sources $S' = \{s(1), s(2) \mid s \in S\}$
- let M the gammoid on D' with sources S'
- set $\mathcal{Y} = \{\{u(1), v(2)\} \mid \{u, v\} \in P\}$
- let $\mathcal{Y}^*$ be $2k$-representative subset of $\mathcal{Y}$ of size $\binom{2k+2}{2} = \mathcal{O}(k^2)$ as given by representative sets lemma
- let $P^* = \{\{u, v\} \mid \{u(1), v(2)\} \in \mathcal{Y}^*\}$

lemma: let X closest to S and of size at most k. then S can reach a pair $\{u, v\} \in P^*$ iff S can reach a pair $\{u, v\} \in P$.

proof sketch for the lemma

lemma: let X closest to S and of size at most k. then S can reach a pair $\{p, q\} \in P^*$ iff S can reach a pair $\{u, v\} \in P$.

proof: S can reach a pair $\{p, q\} \in P^*$ in $D - X$

$\Leftrightarrow$ S can reach $p(1)$ in $D(1) - X(1)$ and $q(2)$ in $D(2) - X(2)$...

$\Leftrightarrow$ $X(1) \cup \{p(1)\}$ and $X(2) \cup \{p(q)\}$ are independent ...

$\Leftrightarrow$ $X(1) \cup X(2) \cup \{p(1), q(2)\}$ is independent for some $\{p, q\} \in P^*$

$\Leftrightarrow$ $X(1) \cup X(2) \cup \{u(1), v(2)\}$ is independent for some $\{u, v\} \in P$

$\Leftrightarrow$ $X(1) \cup \{u(1)\}$ and $X(2) \cup \{v(q)\}$ are independent...

$\Leftrightarrow$ S can reach $u(1)$ in $D(1) - X(1)$ and $v(2)$ in $D(2) - X(2)$...

$\Leftrightarrow$ S can reach a pair $\{u, v\} \in P$ in $D - X$

a corollary

lemma: let X closest to S and of size at most k. then S can reach a pair $\{u, v\} \in P^*$ iff S can reach a pair $\{u, v\} \in P$.

corollary: the instances (D, S, P, k) and (D, S, P^*, k) are equivalent.

proof: suffices to check that (D, S, P^*, k) **yes** implies (D, S, P, k) **yes**

- if (D, S, P^*, k) is **yes** then take a solution X closest to S
- if X would leave some pair $\{u, v\} \in P$ connected to S
- then $\{u(1), v(2)\} \in \mathcal{Y}$ would extend $X(1) \cup X(2)$ in M
- but then some $\{p(1), q(2)\} \in \mathcal{Y}^*$ would extend it too
- implying that $\{p, q\}$ is reachable from S in $D - X$; contradiction

now: storing instance as gammoid (similar to OCT)...

Gammoid representation of pair cuts

given: DPC instance $D = (V, A)$, sources S, $\mathcal{O}(k^2)$ pairs P^*
want: to see how to get this from a gammoid on D with sources S

- let $M = (V(P^*), \mathcal{I})$ gammoid on D with sources S
- guess/try a choice $T \subseteq V(P^*)$ of vertices to disconnect from S s.t. $\{u, v\} \in P^*$ implies $u \in T$ or $v \in T$
- cost to disconnect S from T equals S, T-mincut ...
- ... equals maximum S, T-path packing ...
- ... equals largest independent set $T' \subseteq T$

conclusion: gammoid on D with sources S and sinks $V(P^*)$ provides all the needed information

Digraph Pair Cut – Wrap up

- saw our first "real" application of representative sets
- in essence we have generalized what was known for Vertex Cover
- reduced a given set of constraints (edges/pairs) to a smaller set that could still witness (in-)correctness of proposed solutions
- crucial aspects:
  1. certain (closest) solutions correspond to independent sets
  2. appropriate way of extending such independent sets corresponds to reachable pairs
- final outcome is a compression

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Multiway Cut with deletable terminals

Multiway Cut with deletable terminals (DTMWC)
Input: A graph $G = (V, E)$, a set $T \subseteq V$ of terminals and an integer k.
Output: Can one delete at most k vertices (also terminals) such that each component contains at most one terminal?

- NP-complete from Vertex Cover
- outline of kernelization argument:
  1. identify vertices separating many terminals w.r.t. some solution
  2. formalize this as uniquely extending some independent set
  3. representative sets lemma: get small set containing these vertices

note: easy to reduce to $|T| = \mathcal{O}(k^2)$ and known how to get to $|T| \leq 2k$.
hence we will focus on non-terminal vertices in solutions.

Are minimum solutions independent?

consider gammoid M on $G = (V, E)$ with sources $T \subseteq V$

- let X be a set of k vertices whose deletion separates all terminals T
- if X can be separated from T by some separator X' of size at most $|X|$ then we can replace X by X'
- take X to be closest to T (i.e. X is unique min X, T-separator)
- thus for every $x \in X \setminus T$ the set $X \cup \{x'\}$ is independent[2]

question: can we leverage this fact into a useful application of the representative sets lemma?

note: throughout this part we silently assume that gammoids contain enough sink-only copies of each vertex

[2] recall sink-only copy x', notion of closeness and independence property

Attempt at using representative sets

Idea: Use gammoid M on $G = (V, E)$ with sources T. Know that $\{x'\}$ extends solutions X closest to T (when $x \in X \setminus T$). Does x' show up in representative sets?

Intuition:

- let $M = (U, \mathcal{I})$ be any matroid
- let $\mathcal{Y}$ a family of independent sets and let $Y \in \mathcal{Y}$
- assume that Y uniquely extends some independent set X i.e.
  - $X \cap Y = \emptyset$
  - $X \cup Y$ is independent
  - no other $Y' \in \mathcal{Y}$ fulfills this
- then Y must be contained in every $|X|$-representative subset $\mathcal{Y}^*$ of $\mathcal{Y}$

Attempt at using representative sets

Idea: Use gammoid M on $G = (V, E)$ with sources T. Know that $\{x'\}$ extends solutions X closest to T (when $x \in X \setminus T$). Does x' show up in representative sets?

- let $\mathcal{Y} = \{\{v'\} \mid v \in V \setminus T\}$
- let $\mathcal{Y}^*$ be a k-representative subset of $\mathcal{Y}$
  - for analysis fix some solution X and $x \in X \setminus T$
  - know: $\{x'\}$ extends the independent set X of size at most k
  - but: many other sets $\{v'\}$ can also extend X [3]
  - thus: no guarantee that $\{x'\} \in \mathcal{Y}^*$

problem: $\{x'\}$ indeed extends X but it's not unique in this property
solution?: try perspective that $\{x, x'\}$ extends $X - x$

[3] true for every v reachable from T in $G - X$

Attempt at using representative sets II

Idea II: Use gammoid M on $G = (V, E)$ with sources T. Know that $X \cup \{x'\}$ is independent for solutions X closest to T and $x \in X \setminus T$. Could say that $\{x, x'\}$ extends $X - x$. Does $\{x, x'\}$ show up in representative sets?

- let $\mathcal{Y} = \{\{v, v'\} \mid v \in V \setminus T\}$
- let $\mathcal{Y}^*$ be a $k-1$-representative subset of $\mathcal{Y}$
  - for analysis fix some solution X and $x \in X \setminus T$
  - know: $\{x, x'\}$ extends the independent set $X - x$
  - but: many other sets $\{v, v'\}$ can also extend $X - x$
  - thus: no guarantee that $\{x, x'\} \in \mathcal{Y}^*$

problem: $\{x, x'\}$ indeed extends $X - x$ but it's not unique in this property
solution?: need a stronger property than independence of $X \cup \{x'\}$

Important vertices in a solution to DT-MWC?

given: Graph $G = (V, E)$, terminals $T \subseteq V$, integer k. Solution $X \subseteq V$.
want: Understanding of what makes some vertices in $X \setminus T$ important.

- **idea:** "make a hole in X and see what happens" :-)
- consider $G' = G - (X \setminus \{x\})$ for some $x \in X \setminus T$
- some terminals may be connected in G' as we do not delete x
- assume that three terminals t_1, t_2, t_3 remain connected, then
  - x intersects all paths between these terminals in G'
  - no other vertex-deletion can disconnect all three terminals
  - x has three vertex-disjoint paths from $\{t_1, t_2, t_3\}$
  - no other vertex has three disjoint paths from $\{t_1, t_2, t_3\}$

hope: in this case, x is the unique vertex such that there are $|X| - 1$ paths from T to $X \setminus \{x\}$ plus three paths from T to x

Important vertices in a solution to DT-MWC?

given: Graph $G = (V, E)$, terminals $T \subseteq V$, integer k. Solution $X \subseteq V$.
want: Understanding of what makes some vertices in $X \setminus T$ important.

let $x \in X \setminus T$ and consider $G' = G - (X \setminus \{x\})$

- if x reaches at most one terminal then $X \setminus \{x\}$ is a solution too
- if x reaches two terminals t, t' then $(X \setminus \{x\}) \cup \{t\}$ is a solution too
- else: x reaches at least three terminals in G'

thus in solutions X with minimum $|X \setminus T|$ all $x \in X \setminus T$ separate at least three terminals

goal: prove that $X \cup \{x', x''\}$ is independent (two sink-copies of x)

A Hall-type argument for DT-MWC

let X be a solution with minimum $|X \setminus T|$, and let $x \in X \setminus T$

- make auxiliary bipartite graph $H = ((X \setminus T) \cup \{x', x''\}, T \setminus X, F)$
  - left side: non-terminal solution vertices with two extra copies of x
  - right side: all undeleted terminals
  - let edge $\{x, t\} \in F$ if t reaches x in $G - (X - x)$
- assume that no matching saturates $(X \setminus T) \cup \{x', x''\}$
- by Hall's Thm there is $Q \subseteq (X \setminus T) \cup \{x', x''\}$ with $|N_H(Q)| < |Q|$
- let $Q' = Q \setminus \{x', x''\}$ then
  - $|N_H(Q')| \leq |N_H(Q)| < |Q| \leq |Q'| + 2$
  - i.e. $|N_H(Q')| \leq |Q'| + 1$
- **exercise:** deleting all but one terminal from $N_H(Q')$ instead of Q' gives a solution and has fewer non-terminal deletions

conclusion: have matching saturating $(X \setminus T) \cup \{x', x''\}$

Using the matching

conclusion: have matching saturating $(X \setminus T) \cup \{x', x''\}$

- each $y \in X \setminus T$ has a private terminal $t \in T \setminus X$ (from matching) that can reach y without passing other vertices of X
- for x we even have three such terminals
- crucially all paths for reaching X are in separate components
- thus matching gives $|X| + 2$ disjoint paths from T to $X \cup \{x', x''\}$
- thus $X \cup \{x', x''\}$ is independent in gammoid on G with sources T

conclusion: if X has minimum $|X \setminus T|$ then $X \cup \{x', x''\}$ is independent for all $x \in X \setminus T$

Attempt at using representative sets III

Idea III: Know that $X \cup \{x', x''\}$ is independent for solutions X with minimum $|X \setminus T|$ and $x \in X \setminus T$. Thus $\{x, x', x''\}$ extends $X - x$. Does $\{x, x', x''\}$ show up in representative sets?

- let $\mathcal{Y} = \{\{v, v', v''\} \mid v \in V \setminus T\}$
- let $\mathcal{Y}^*$ be a $k-1$-representative subset of $\mathcal{Y}$
  - for analysis fix some solution X with min $|X \setminus T|$ and $x \in X \setminus T$
  - know: $\{x, x', x''\}$ extends independent set $X - x$ of size at most $k - 1$
  - are there any other sets $\{v, v', v''\}$ that extend $X - x$?

Attempt at using representative sets III – intermission

let X be solution as before

- assume that $\{v, v', v''\}$ extends $X - x$ for some $v \neq x$
- must have paths from three terminals t_1, t_2, t_3 to v that overlap only in v and avoid $X - x$
- only one of the paths, say t_3, v-path, can contain $x \neq v$
- but then we get t_1, t_2-path avoiding both x and $X - x$ by combining t_1, v- and t_2, v-paths
- contradiction to X being a solution

Attempt at using representative sets III – completion

Idea III: Know that $X \cup \{x', x''\}$ is independent for solutions X with minimum $|X \setminus T|$ and $x \in X \setminus T$. Thus $\{x, x', x''\}$ extends $X - x$. Does $\{x, x', x''\}$ show up in representative sets?

- let $\mathcal{Y} = \{\{v, v', v''\} \mid v \in V \setminus T\}$
- let $\mathcal{Y}^*$ be a $k-1$-representative subset of $\mathcal{Y}$
  - for analysis fix some solution X with min $|X \setminus T|$ and $x \in X \setminus T$
  - know: $\{x, x', x''\}$ extends independent set $X - x$ of size at most $k - 1$
  - no other set $\{v, v', v''\}$ can extend $X - x$
  - hence $\{x, x', x''\}$ must be in $\mathcal{Y}^*$

conclusion: if X is a solution of size at most k with minimum $|X \setminus T|$ then $\mathcal{Y}^*$ contains $\{x, x', x''\}$ for all $x \in X \setminus T$.
in other words: we get $X \subseteq T \cup V(\mathcal{Y}^*)$

Checking the size of $\mathcal{Y}^*$

Lemma: Let $M = (E, \mathcal{I})$ be a represented matroid and let $\mathcal{Y}$ be a collection of independent sets, each of size s.
There is a set $\mathcal{Y}^*$ of size at most $\binom{r+s}{s}$ that is r-representative for $\mathcal{Y}$.

- we have $\mathcal{Y} = \{\{v, v', v''\} \mid v \in V \setminus T\}$ i.e. $s = 3$
- we request $k-1$-representative subset $\mathcal{Y}^*$, i.e. $r = k - 1$
- thus get $|\mathcal{Y}^*| \leq \binom{r+s}{s} = \mathcal{O}(k^3)$

recall: regarding representation of M

- efficient randomized algorithm for representation of a gammoid
- exponentially small error
- size of representation "does not matter" (beyond P-time)

Outline of kernelization

given graph $G = (V, E)$, terminals $T \subseteq V$ and integer k

- use gammoid $M = (V \cup V' \cup V'', \mathcal{I})$ on G with sources T
- let $\mathcal{Y} = \{\{v, v', v''\} \mid v \in V\}$
- representative sets lemma: get $k-1$-representative subset $\mathcal{Y}^* \subseteq \mathcal{Y}$
- **know:** there is a solution $X \subseteq T \cup V(\mathcal{Y}^*)$ (if one exists)
- can make $V \setminus (T \cup V(\mathcal{Y}^*))$ undeletable:
  - add shortcut edges for all paths with undeletable internal vertices
  - remove $V \setminus (T \cup V(\mathcal{Y}^*))$ from G to get G'

theorem: randomized polynomial kernelization for DT-MWC with $\mathcal{O}(k^3)$ vertices and one-sided exponentially small error (only false negatives).

DTMWC – Wrap up

- different style of application for representative sets (+ gammoids)
- before (Digraph Pair Cut) any representative set was good enough
- now we used representative sets to identify important vertices:
  1. identify large independent set related to certain vertices in solution
  2. set up tuples consisting of vertices + their sink-only copies
  3. exhibit independent set such that only the tuple of a desired vertex can extend it
  4. thus all important vertices are reflected in representative set
- this leads to a reduction-rule based kernel for DT-MWC
- randomization only needed for gammoid representation
- size independent of representation and error

Conclusion

just namedropping a few things that you might/should remember:

- how to get a gammoid that answers all 2-way cuts on "terminals" X
- (closest) S, T-cuts in D vs. independent sets of gammoid on (D, S)
- representative sets & extending independent sets
- representative sets lemma
  - using any representative subset (as for digraph pair cut)
  - enforcing "survival" of important vertices in representative set